Crash-Starting the (Dual) Simplex Method

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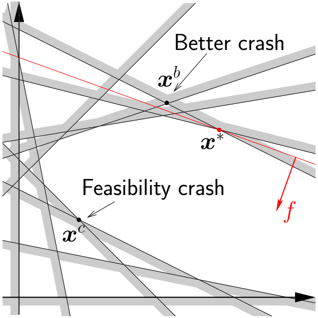
Overview

A linear programming problem is

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$



$$\mathbf{x}^0 \xrightarrow{\text{Crash}} \mathbf{x}^k \xrightarrow{\text{Crossover}} \mathbf{x}^b \xrightarrow{\text{Simplex}} \mathbf{x}^*$$

Motivation

- A good starting basis is essential for the solution of large problems
- The Idiot crash is a heuristic aiming to improve feasibility
- Implemented by John Forrest in CLP
- Seeking primal feasibility prior to primal simplex

$$\mathbf{x}^0 \xrightarrow{\text{Crash}} \mathbf{x}^k \xrightarrow{\text{Crossover}} \mathbf{x}^b \xrightarrow{\text{Simplex}} \mathbf{x}^*$$

General idea

The idea is to solve many small problems instead of one big one

In turn

- Fix all components of $\mathbf{x}$ except x_1
- Fix all components of $\mathbf{x}$ except x_2
- Fix all components of $\mathbf{x}$ except x_3
- ...

During the execution parameters are adjusted according to progress

The Idiot Crash

The Idiot Crash structure

Initialize $\mathbf{x}^0$

For $I \in 1..\text{nMajor}$

For $i \in 1..\text{nMinor}$

$\forall j \in \{1..n\} \quad \min_{x_j} g(\mathbf{x})$

end

end

$$\min_{x_j} g(\mathbf{x})$$

$$\Updownarrow$$

Minimize a quadratic function in one variable

The function minimized

Two goals:

$$\min \mathbf{c}^T\mathbf{x} \quad \text{and} \quad \sum_i e_i(\mathbf{x})^2$$

where $e_i(\mathbf{x})$ is the infeasibility of constraint i.

The function minimized for each column:

$$\min_{x_j} \quad \mu\tilde{\mathbf{c}}^T\mathbf{x} + \sum_{i\in 1..m} e_i(\mathbf{x})^2 = g(\mathbf{x})$$

$$\Updownarrow$$

$$\min_{x_j} \quad \mu\left(c_j + \sum_i a_{ij}\lambda_i\right)x_j + \sum_{i\in 1..m} e_i(\mathbf{x})^2 = g(\mathbf{x})$$

The Idiot Crash

The Idiot Crash structure

Initialize $\mathbf{x}^0$, μ^0, $\boldsymbol{\lambda}^0 = \mathbf{0}$
For $l \in 1..\text{nMajor}$
    For $i \in 1..\text{nMinor}$

$$\forall j \in \{1..n\} \quad \min_{x_j} g(\mathbf{x}) = \mu \left(c_j + \sum_i a_{ij}\lambda_i \right) x_j + \sum_{i \in 1..m} e_i(\mathbf{x})^2$$

    end
  Choose new μ or $\boldsymbol{\lambda}$
end

Every few major iterations μ is reduced

Parameters

At the end of each major iteration the values of $\boldsymbol{\lambda}$ or μ are re-adjusted:

- Start with μ^0, $\boldsymbol{\lambda}^0 = \mathbf{0}$, nMinor $= 2$
- At the end of each major iteration:
  - Until 10% reduction in primal infeasibility set $\mu^{k+1} = \mu^k/10$
- Set nMinor $= 105$
  - The first major iteration with a new μ, $\boldsymbol{\lambda}$ is unchanged
  - The next few major iterations set $\lambda_i^{k+1} = \frac{1}{\mu} e_i(\mathbf{x}^k)$
  - Reduce μ and repeat

Further details

- During initial major iterations μ is bigger so $\boldsymbol{\lambda}$ ensures reducing primal infeasibility has enough weight

$$\min_{x_j} \quad \mu \left(c_j + \sum_i a_{ij}\lambda_i \right) x_j + \sum_{i \in 1..m} e_i(\mathbf{x})^2$$

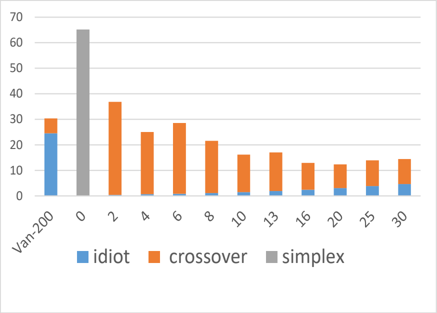
- Cycles through all columns
  - Random starting point
  - Alternating direction
- Early termination of a major iteration is possible
  - Condition on moving average of expected progress

Performance

- Generally useless
  - Single variable change limitation
- Solves Quadratic Assignment Problems (QAPs)
  - Combinatorial optimization problem, special case of facility location
  - Well known for being hard to solve
  - The Idiot crash reduces solution time significantly

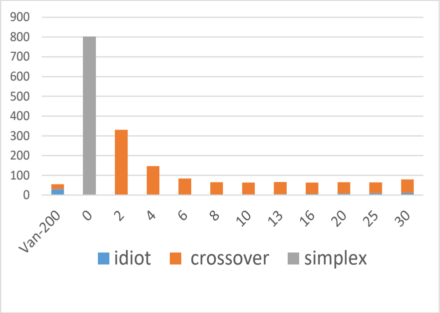
Quadratic Assignment Problems

QAP12
Solution time (s)



nMajor

QAP15
Solution time (s)



nMajor

Idiot alone

$$\begin{aligned} \min \quad & f_P = \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \qquad (P) \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \max \quad & f_D = \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & A^T\mathbf{y} \neq \mathbf{c} \qquad (D) \end{aligned}$$

Idiot applied to (P)

- Primal feasible
- Near-optimal objective $f_P \geq f^*$

Idiot applied to (D)

- Dual feasible
- Objective $f_D \leq f^*$

Observations

				Time (s)		
Problem	LB	UB	f^*	Bounds	Idiot+Crossover	Simplex
qap08	141.39	202.57	203.50	0.55	0.61	1.80
qap12	481.50	523.60	522.89	3.85	12.64	65.21
qap15	950.05	1042.05	1040.99	11.53	64.48	803.30

- QAPs require integer solution to linearization
- Fast bounds on f^* accelerate branch & bound algorithms

Conclusion

- Finding a good starting basis beneficial but difficult
- The Idiot crash has limitations
  - Sensitive to parameters
  - Can have disastrous effects
  - Single variable change
- Effective on some problems
- Scope for accelerating B&B